

# Example of a non-Gaussian $n$ -point bifurcation for stochastic Lévy flows

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## Abstract

We explore the elementary observation that a Markov chain with values in a finite space  $M$  with  $|M| = m$ ,  $m \geq 2$ , has many different extensions to a compatible  $n$ -point Markov chain in  $M^n$ , for all  $1 < n \leq m$ . Embedding this phenomenon into the context of stochastic Lévy flows of diffeomorphisms in Euclidean spaces, we introduce the notion of an  $n$ -point bifurcation of a stochastic flow. Roughly speaking a  $n$ -point bifurcation takes place, when a small perturbation of the stochastic flow does not change the characteristics at lower level  $k$ -point motions,  $k < n$ , but does change at the level of  $n$ -point motion. We illustrate this phenomenon with an example of an  $n$ -point bifurcation, with  $n \geq 3$ . In addition, we present an algorithm for the detection of the precise level of an  $n$ -point bifurcation and a combinatorial formula for the dimension of the vector space of compatible extensions for flows of mappings on  $M$ .

Keywords: stochastic dynamics, stochastic bifurcation,  $n$ -point process, stochastic Lévy flow, Markovian process, algorithmic detection of bifurcations, Stratonovich SDE, canonical Marcus SDE.

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## 1 Introduction

It is well-known for Brownian flows that the law of the stochastic flow is uniquely determined by the laws of the family of the corresponding 1-point and 2-point motions, see Kunita [10]. This behavior is due to the Gaussian nature of the marginal laws and can be read from the structure of the infinitesimal generator. General Lévy flows do not exhibit this behavior. In this article we construct a simple example, which motivates us to coin the notion of a general stochastic  $n$ -point bifurcation.

We explore the elementary observation in the context of stochastic flows that a Markov chain with values in a finite space  $M = \{1, \dots, m\}$ , with  $m > 1$ , has many different extensions to a compatible  $n$ -point Markov chain in  $M^n$ , for all  $1 < n \leq m$ . Each extended process in  $M^n$  is called an  $n$ -point lift of the original 1-point motion in  $M$ . Compatibility here means that any projection of the process in  $M^n$  into an embedded  $M^k$ , with  $k \leq n$  leads to a Markov chain in  $M^k$  with the same law, such that, eventually, after any  $(n-1)$  projections, the original law of the 1-point motion in  $M$  is recovered. Two different lifts to  $n$ -points can still have laws that coincide in the  $k$ -point process with  $1 < k < n$ . This means that the laws of these two lifts can only be distinguished one from another if an observer studies the statistics of the  $j$ -point process for  $j > k$  sufficiently large.

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We recall that given a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , a (time-homogeneous, measurable) stochastic flow in a Polish space  $\mathcal{M}$  is a family of parametrized measurable mappings  $\varphi = \{\varphi_t(\omega, \cdot) : \mathcal{M} \rightarrow \mathcal{M} \mid t \geq 0, \omega \in \Omega\}$  such that  $\varphi_0(\omega) = Id_{\mathcal{M}}$  for all  $\omega \in \Omega$  and  $\varphi_{s+t}(\omega, \cdot) = \varphi_s(\theta_t \omega, \cdot) \circ \varphi_t(\omega, \cdot)$  for all  $0 \leq s \leq t$  and  $\omega \in \Omega$ . The time variable  $t$  here is considered either in  $\mathbf{N} \cup \{0\}$  or in  $\mathbf{R}_{\geq 0}$  and the shift  $\theta_t : \Omega \rightarrow \Omega$  is an ergodic measurable preserving transformation. The  $n$ -point motion induced by a stochastic flow  $\varphi$  in  $\mathcal{M}^n$  is given by  $(x_1, \dots, x_n) \mapsto (\varphi_t(x_1), \dots, \varphi_t(x_n))$ . A simple example of stochastic flow with discrete time is the composition of a random i.i.d. sequence  $(\xi_n)_{n \geq 0}$  of mappings in  $\mathcal{M}$ . Then, for a positive integer  $t$ , the flow  $\varphi_t = \xi_t(\omega) \circ \dots \circ \xi_2(\omega) \circ \xi_1(\omega)$ .

In the context of discrete Markovian dynamics in  $M$ , there are three standard approaches on studying the extensions to an  $n$ -point version. The first approach consists in restricting to the random dynamics generated by composition of sequences of random i.i.d. permutations in  $M$  (stochastic flow of bijections). This approach, although quite restrictive, is particularly interesting since it allows easily to embed the results from the finite space  $M$  into a flow of diffeomorphisms in an Euclidean space: just use an appropriate choice of the vector fields and Lévy noise involved in a Stratonovich SDE for general semimartingales of Markus type, as in Kurtz, Pardoux and Protter [11] (see Section 2). The second approach is again a stochastic flow, but here one allows for the composition of random mappings from  $M$  to  $M$ , which permits the coalescence of particles. As in the bijective approach, the diagonal is still invariant (in fact, an attractor here). Finally, the third approach consists in considering a Markovian lift independently of any flow in  $M$ , hence one misses many symmetries in the transition probabilities in  $M^k$ , moreover, (sub-)diagonals may no longer be invariant. Think, for instance, of the simplest example:  $M = \{1, 2\}$  with all possible transition probabilities equal to  $1/2$  and a lifted process in  $M^2 = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$  with all the transition probabilities given by  $1/4$ . This dynamics in  $M^2$  is clearly not generated by a flow of mappings.

In this article we focus on the dynamics generated by stochastic flows, i.e. in the first and second approaches. The main motivation of this choice is that it allows us to introduce the notion of an  $n$ -point bifurcation illustrated by the example of Section 2 and 3.

## 1.1 The notion of an $n$ -point bifurcation

Given invariant measures for the  $n$ -point process, the invariant measures in lower  $k$ -point processes,  $k < n$ , can be recovered using the following well expected result:

**Proposition 1.** *For  $1 \leq k \leq n$  consider the projection*

$$p_k^n : M^n \rightarrow M^{n-1}, \quad (x_1, \dots, x_n) \mapsto (x_1, \dots, \hat{x}_k, \dots, x_n),$$

*and let  $\mu$  be an invariant measure of a Markov process  $X$  in  $M^n$ . If  $p_k^n(X)$  is also a Markov process, then the induced measure  $(p_k^n)_* \mu$  is an invariant measure for the process  $p_k^n(X)$  in  $M^{n-1}$ .*

*Proof.* For convenience we drop the superscript  $n$  whenever possible. Let  $P(x, A)$  be the family of transition probabilities of the process  $X$  in  $M^n$  for  $x \in M^n$  and subsets  $A \subset M^n$ . The fact that the projection  $p_k(X)$  generates a Markov process in  $M^{n-1}$  means that the transition probabilities in  $M^{n-1}$ , denoted by  $\bar{P}(p_k(x), B)$ , is well defined for any  $B \subset M^{n-1}$  and it is given by

$$\bar{P}(p_k(x), B) = P(x, p_k^{-1}(B))$$

for all  $x \in M^n$ . Now, by the induced measure theorem:

$$\begin{aligned}
(p_k)_*\mu(B) = \mu(p_k^{-1}(B)) &= \int_{M^n} P(x, p_k^{-1}(B)) d\mu(x) \\
&= \int_{M^{n-1}} P(p_k^{-1}(y), p_k^{-1}(B)) d(p_k)_*\mu(y) \\
&= \int_{M^{n-1}} \bar{P}(y, B) d(p_k)_*\mu(y).
\end{aligned}$$

□

*The notion of an  $n$ -point bifurcation:* Classically, in dynamical systems, a bifurcation occurs when, changing a parameter of the flow, the topology of the support of an invariant measure is affected somehow, normally splitting into two or more disjoint invariant domains. This is well known and well studied for deterministic dynamical systems, where the precise definition is based on breaking local topological equivalences of the flows (among many classical references see, e.g. Katok and Hasselblatt [8] and references therein). For stochastic systems generated by Itô-Stratonovich equations, the bifurcation is mostly considered as a change of sign of the Lyapunov exponent, see e.g. L. Arnold [2], [3]. These two situations have in common the fact that they are observing a breaking in the topology of the support of invariant measures, but at different levels: In the deterministic case, the invariant measures are considered in  $M^1$ , with trivial extension to  $M^n$  as the respective product measure; in the stochastic case, the sign of the Lyapunov exponents points to properties of the invariant measures in  $M^2$ . See the explicit example by Baxendale [4], where a bifurcation happens for Brownian motions in the torus: i.e. the top Lyapunov exponent change the sign but the law of the 1-point motion is not affected, see also [5].

Going further from these 1 and 2-point phenomena to  $n$ -point motions, we propose the following natural extension of the definition of bifurcation for more general stochastic flows in Polish spaces:

**Definition 1.** Let  $\varphi^\varepsilon$  be a family of stochastic flows acting on a Polish space  $\mathcal{M}$  indexed by a parameter  $\varepsilon$  with values in an interval.

We say that  $\varphi^\varepsilon$  exhibits an  $n$ -point bifurcation,  $n \geq 2$ , at a parameter value  $\varepsilon_0$  if it satisfies the following

1. The mapping  $\varepsilon \mapsto \varphi^\varepsilon$  is continuous with respect to the weak topology at all values  $\varepsilon$  except  $\varepsilon_0$ .
2. There exists an invariant measure  $\mu^{\varepsilon_0}$  with respect to  $\varphi^{\varepsilon_0}$  such that for any  $\varepsilon > \varepsilon_0$  there exists an invariant measure  $\mu^\varepsilon$  with respect to  $\varphi^\varepsilon$  on  $\mathcal{M}^n$  satisfying that

$$\text{supp}(\mu^{\varepsilon_0}) \text{ is not homeomorphic to } \text{supp}(\mu^\varepsilon),$$

and there exists a projection  $p_k^n$  defined in Proposition 1 such that

$$\text{supp}((p_k^n)\mu^{\varepsilon_0}) = \text{supp}((p_k^n)\mu^\varepsilon).$$

□

A first example is given in Section 2. Since each invariant measure on  $\mathcal{M}^1$  can have many lifts to invariant measures in higher levels  $\mathcal{M}^k$ , these lifts can exhibit more than one bifurcation, at different levels  $k$ . Moreover, the same bifurcation on  $k$ -point motion can have projections into different invariant measures on  $\mathcal{M}^1$  (depending on the sequence of projections). This is going to be illustrated in Section 3.1.

As for comparison (following Kunita [10]), we consider a homogeneous Brownian flow  $\varphi$  in the group of diffeomorphisms in an Euclidean space, with the infinitesimal mean

$$b(x) = \lim_{h \rightarrow 0+} \frac{1}{h} (\mathbb{E} [\varphi_h(x)] - x), \quad \forall x \in \mathbb{R}^d,$$

and the infinitesimal covariance

$$a(x, y) = \lim_{h \rightarrow 0+} \frac{1}{h} (\mathbb{E} [(\varphi_h(x) - x)(\varphi_h(y) - y)^*]), \quad \forall x, y \in \mathbb{R}^d$$

whenever defined. Given certain regularity conditions on this parameters (satisfied for instance by flows of SDE generated by smooth vector fields with bounded derivatives) the law of  $\varphi$  in the group of diffeomorphisms is determined by  $a(x, y)$  and  $b(x)$  [10, Thm. 4.2.5, p. 126]. In other words, the law of a (homogeneous) Brownian flow (hence the law of its  $n$ -point motion, with  $n \geq 2$ ) is fully determined by the laws of its 1-point motion and its 2-point motion. This theorem tells us that classical stochastic flows for SDE generically do not furnish the richness of flows differing only on higher  $n$ -point motion, with  $n > 2$ , as we are looking for.

The problem we are addressing here is somehow related to the recent results by Jost, Kell and Rodrigues [7], where they study conditions under which the transition probabilities (1-point motion) in a manifolds can be represented by families of random maps. In the same article, they consider further conditions for regularity and representations by diffeomorphisms. For this kind of problem, where measurability of the flows is considered we refer to Kifer [9] and Quas [12].

The flows we are interested here are also related to the flow of measurable mappings of Le Jan and Raimond [13] (see also [14]) in the following sense: their flows are constructed from a family of Feller compatible semigroups in  $C(M^n)$ ,  $n \geq 1$ , which preserves the diagonal. They are also constructed based on the observation of the statistics of the  $n$ -point motion, for  $n \geq 1$ .

*Flow of bijections in  $M$ :* The celebrated Birkhoff-von Neumann theorem says that  $n \times n$  bi-stochastic matrices lay in the convex hull generated by the  $m!$  matrices of permutations in  $\{1, 2, \dots, m\}$ . This convex set is called the Birkhoff polytopes  $P_n$ . There are several proofs of this theorem in the literature. For a simple and elementary proof see e.g. Mirsky [15]. This theory has many interesting application, and although already very studied, still has some good open problems, see e.g. Pak [16]. For instance, despite its relevance, for higher dimensional Birkhoff polytopes  $P_n$ , there is no formulae for the volume of  $P_n$ ; only recently an asymptotic formula was obtained by Canfield and McKay [6]. In the context of our article, for random dynamics generated by i.i.d. random mappings, it means that a stochastic flow in  $M$  is a flow of permutations if and only if the matrices of transition probabilities of 1-point motion is bi-stochastic. Moreover, in the Birkhoff polytope language, what we are exploring in this article is the fact that, in general, except for elements in the wedges of the polytope  $P_n$ , the bi-stochastic matrices has a non-unique representation as a linear combination of the vertices of  $P_n$  (in fact,  $P_n$  is contained in a  $(m-1)^2$ -dimensional subspace and has  $m!$  vertices).

The main objectives of this article are the following:

1. To illustrate the  $n$ -point bifurcation of Definition 1 beyond the well known examples mentioned above (for  $n = 1$  and  $n = 2$ ). Our example starts with a finite space  $M$  which exhibits  $n$ -point bifurcation, for  $n \geq 3$ . This bifurcation is then embedded in an Euclidean space as a flow of diffeomorphisms generated by a generalized Stratonovich equation (Markus type as in [11]) driven by a Lévy noise .
2. Given two different stochastic flows in a finite state space with the same transition probabilities in 1-point motion, we present an algorithm to establish what is the minimal positive integer  $k$ , such that the transition probabilities are different, or yet, the minimal positive integer  $k$  such that the invariant measures have topologically non homeomorphic support.
3. We present a formula for the dimension of the space of all possible vector space of probabilities on the mappings from  $M$  to  $M$  such that the corresponding stochastic flow (generated by composition of i.i.d.) satisfies a prescribed  $k$ -point transition probabilities, with  $0 \leq k < m$ .

Items (1),(2) and (3) above are treated in Sections 2, 3 and 4 respectively.

## 2 Example of an $n$ -point bifurcation with $n \geq 3$

The purpose of this section is to show an example for the higher order  $n$ -point bifurcation, which does not change the characteristics (transition probabilities) in lower order  $k$ -points for  $k < n$ . We construct a discrete stochastic flow in  $M = \{1, 2, 3, 4, 5, 6\}$  generated by the composition of i.i.d. random bijections with the following property: When the law of the random bijections are perturbed by a parameter  $\epsilon \in [0, 1]$  in a certain direction, the invariant measures of the 6-point motion have different supports for  $\epsilon = 0$  and for  $\epsilon \in (0, 1]$  but the laws of the corresponding 1-point and 2-point motions remain constant with respect to  $\epsilon$ . Hence, this family of stochastic flows indexed by  $\epsilon$  exhibits a bifurcation for the  $n$ -point motions for  $3 \leq n \leq 6$ . This result is embedded into flows in  $\mathbf{R}^7$  in the next section.

**The basic construction:** We adopt the notation  $f_{i_1 i_2 i_3 i_4 i_5 i_6}$  for the function

$$(1, 2, 3, 4, 5, 6) \mapsto (i_1, i_2, i_3, i_4, i_5, i_6).$$

with  $i_1, \dots, i_m \in M$ . The following double notation is convenient for our example:

$$(1, 2, 3, 4, 5, 6) = ((1, 2), (3, 4), (5, 6)) = (a, b, c)$$

and denote by  $\bar{a}, \bar{b}$  and  $\bar{c}$  the flips of each double entry, for example  $(\bar{a}, \bar{b}, c) = (2, 1, 4, 3, 5, 6)$ .

Consider the group

$$G = \{f_{abc}, f_{\bar{a}bc}, f_{a\bar{b}c}, f_{ab\bar{c}}, f_{\bar{a}\bar{b}c}, f_{a\bar{b}\bar{c}}, f_{\bar{a}b\bar{c}}, f_{\bar{a}\bar{b}\bar{c}}\}$$

with multiplication given by the composition, and its proper subgroup

$$H = \{f_{abc}, f_{\bar{a}bc}, f_{ab\bar{c}}, f_{a\bar{b}\bar{c}}\}.$$

The 6-point motion of a flow  $\varphi^0$  generated by composition of i.i.d. bijections with law concentrated on this subgroup, say

$$\frac{1}{4} [\delta_{f_{abc}} + \delta_{f_{\bar{a}bc}} + \delta_{f_{ab\bar{c}}} + \delta_{f_{a\bar{b}\bar{c}}}] , \tag{2.1}$$

has random trajectories with the following property: for each initial condition in  $M^6$  the corresponding random orbit of the process is concentrated on at most 4 points out of  $6^6$  possible elements of  $M^6$ . For an initial condition which does not belong to any subdiagonal (i.e. such that their entries are all different from each other), the support of the invariant measure is concentrated on exactly 4 elements. On the other hand, the orbits of elements outside any sub-diagonal of the 6-point motion  $\varphi^\varepsilon$  generated by the  $\varepsilon$ -perturbation in the law, with  $\varepsilon > 0$ ,

$$\frac{1}{4} [\delta_{f_{abc}} + \delta_{f_{\bar{a}\bar{b}\bar{c}}} + \delta_{f_{a\bar{b}\bar{c}}} + \delta_{f_{a\bar{b}\bar{c}}}] + \frac{\varepsilon}{4} [\delta_{f_{\bar{a}bc}} + \delta_{f_{a\bar{b}c}} + \delta_{f_{a\bar{b}c}} + \delta_{f_{\bar{a}\bar{b}\bar{c}}} - \delta_{f_{abc}} - \delta_{f_{\bar{a}\bar{b}\bar{c}}} - \delta_{f_{a\bar{b}\bar{c}}} - \delta_{f_{a\bar{b}\bar{c}}}], \quad (2.2)$$

has invariant measures supported on exactly 8 elements. Moreover, one easily checks by inspection that, due to appropriate cancellations, the transition probability of jumps from a pair of points to any other pair of points does not depend on  $\varepsilon$ , i.e the law in  $M^2$  is constant. The same happens for the law in  $M^1$ .

The splitting on the number of connected components of the support of the invariant measure implies that there must exist an  $n$ -point bifurcation for  $3 \leq n \leq 6$ . In the next section we shall construct an algorithm to find out exactly at which level  $n$  the bifurcation occurs.

**Embedding the  $n$ -point bifurcation into flows of diffeomorphisms:** Suppose  $\nu$  is a probability measure on the set  $S = M \rightarrow M$  of all mappings of  $M$  to itself. Suppose  $(\xi_n)_{n \in \mathbb{N}}$  are i.i.d. in  $S$  with distribution  $\nu$  and define

$$R_n := \xi_n \circ \xi_{n-1} \circ \cdots \circ \xi_1.$$

That is,  $R_n$  is the discrete time random walk of the semigroup  $S$  generated by the probability measure  $\nu$ . To obtain a continuous time version, consider  $Y_t := R_{\pi_t}$ , where  $(\pi_t)_{t \geq 0}$  is an independent standard Poisson process. For  $f \in S$  let  $K(f)$  denote the orthogonal linear mapping of  $\mathbb{R}^m$  to itself, which sends the canonical basis vectors  $e_1, \dots, e_m$  to  $e_{f(1)}, \dots, e_{f(m)}$  respectively. Then  $X_t = K(Y_t)$  is a Lévy flow of linear mappings of  $\mathbb{R}^m$  to itself. If  $\nu$  is supported on the subset of  $S$  consisting of invertible mappings then  $X_t$  is a Lévy flow of diffeomorphisms. For further properties on the representation of Markov chains by sequences of random maps we refer to [9].

Again, as in the discrete case, the laws of the 1-point motion and 2-point motion does not depend on the parameter  $\varepsilon$ . But for  $\varepsilon = 0$  the support of the invariant probability measure for the 6-point motion which includes, say the element  $(e_1, e_2, e_3, e_4, e_5, e_6)$  is given by the set of 4 elements (4 connected components)

$$\left\{ (e_{f(1)}, e_{f(2)}, e_{f(3)}, e_{f(4)}, e_{f(5)}, e_{f(6)}), \text{ for all } f \in H \right\},$$

which splits, for  $\varepsilon > 0$  into the 8 connected components

$$\left\{ (e_{f(1)}, e_{f(2)}, e_{f(3)}, e_{f(4)}, e_{f(5)}, e_{f(6)}), \text{ for all } f \in G \right\}.$$

Hence, for this stochastic flow of diffeomorphisms, an  $n$ -point bifurcation holds at  $\varepsilon = 0$  with  $3 \leq n \leq 6$ . This example illustrates the well known fact that the Kunita's result mentioned in the Introduction [10, Thm. 4.2.3] for stochastic classical Brownian flows does not hold for Lévy flows.

### 3 Detection of the level of an $n$ -point bifurcation

As before, consider the finite space  $M = \{1, \dots, m\}$  with  $m \geq 2$ . The purpose of this section is to answer the following question: given two transition probabilities at the  $m$ -point motion, whose

projections coincide at level of  $k$ -point,  $k < m$ . What is the lowest level of projections on  $n$ -points,  $k < n \leq m$  where they differ? Below, we set up an appropriate algorithm, which we apply to show that the example of Section 2 exhibits an  $n$ -point bifurcation, for  $n = 3$  and  $n = 5$ .

### 3.1 The background and the algorithm

For each  $1 \leq n \leq m$ , let  $A^{(n)}$  be the  $m^n \times m^n$  right stochastic matrix (i.e. the sum of rows are 1), whose entries are the transition probabilities among the elements of  $M^n$ , in the lexicographical order. Since we are dealing with flows, the transition probabilities of  $A^{(n-1)}$  can be obtained from the projections  $p_r^n$  for any  $r \in \{1, \dots, n\}$  defined in Proposition 1. More precisely, for all  $1 \leq r \leq n$  and  $(i_1, \dots, i_n), (j_1, \dots, j_n) \in M^n$  we have

$$A_{(i_1, \dots, \hat{i}_r, \dots, i_n), (j_1, \dots, \hat{j}_r, \dots, j_n)}^{(n-1)} = \sum_{j_r \in M} A_{(i_1, \dots, i_r, \dots, i_n), (j_1, \dots, j_r, \dots, j_n)}^{(n)}, \quad (3.1)$$

where  $(i_1, \dots, \hat{i}_r, \dots, i_n)$  denotes the vector  $(i_1, \dots, i_{r-1}, i_{r+1}, \dots, i_n) \in M^{n-1}$ .

This procedure defines a projection  $\pi_n$  of  $A^{(n)}$  onto  $A^{(n-1)}$ . For each fixed  $r$  and  $i_r \in M$ , there exists a pair of matrices  $(P_{n-1}, Q_{n-1})$  which, according to formula (3.1), satisfies the equation

$$A^{(n-1)} = P_{n-1} A^{(n)} Q_{n-1},$$

where  $P_{n-1}$  is a  $(m^{n-1} \times m^n)$ -dimensional matrix with zero entries except exactly a unique entry 1 in each row and  $Q_{n-1}$  is an  $(m^n \times m^{n-1})$ -dimensional matrix with zero entries except, again, a unique 1 in each row.

**Example:** For  $m = 2$ , assuming compatibility of the matrix of probability transitions  $A^{(2)}$  in  $M^2$ , we calculate  $P_1 \in M_{2 \times 4}$  and  $Q_1 \in M_{4 \times 2}$  for different choices of  $(r, i_r)$ . For  $r = 1$  and  $i_r = 1$  we obtain

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} A_{11,11} & A_{11,12} & A_{11,21} & A_{11,22} \\ A_{12,11} & A_{12,12} & A_{12,21} & A_{12,22} \\ A_{21,11} & A_{21,12} & A_{21,21} & A_{21,22} \\ A_{22,11} & A_{22,12} & A_{22,21} & A_{22,22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix}.$$

For  $r = 1$  and  $i_r = 2$ :

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A_{11,11} & A_{11,12} & A_{11,21} & A_{11,22} \\ A_{12,11} & A_{12,12} & A_{12,21} & A_{12,22} \\ A_{21,11} & A_{21,12} & A_{21,21} & A_{21,22} \\ A_{22,11} & A_{22,12} & A_{22,21} & A_{22,22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix}.$$

Instead, for  $r = 2$  and  $i_r = 1$ :

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} A_{11,11} & A_{11,12} & A_{11,21} & A_{11,22} \\ A_{12,11} & A_{12,12} & A_{12,21} & A_{12,22} \\ A_{21,11} & A_{21,12} & A_{21,21} & A_{21,22} \\ A_{22,11} & A_{22,12} & A_{22,21} & A_{22,22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix}.$$

Note that in all these examples permuting simultaneously lines of  $P_1$  and columns of  $Q_1$  leaves the product invariant.



For higher levels, with  $m > 2$ , fixing  $r = i_r = 1$ , thanks to the lexicographical order on the entries of the matrices  $A^{(n)}$  we have a standard way of performing the projections of transition probabilities, using that:

$$P_{n-1} = \begin{pmatrix} Id_{m^{n-1}} & \vdots & 0 & \vdots & \dots & \vdots & 0 \end{pmatrix}_{m^{n-1} \times m^n}$$

where ‘0’ above represents the null  $(m^{n-1})$ -square matrices. And

$$Q_{n-1} = \begin{pmatrix} Id_{m^{n-1}} \\ Id_{m^{n-1}} \\ \vdots \\ Id_{m^{n-1}} \end{pmatrix}_{m^n \times m^{n-1}}. \quad (3.2)$$

Proposition 1 implies that given a (left) eigenvector  $v_n \in \mathbb{R}^{m^n}$  of  $A^{(n)}$ , its projection  $v_{n-1}$  in  $\mathbb{R}^{m^{n-1}}$  is again an eigenvector of  $A^{(n-1)}$ . More precisely we have the following proposition.

**Proposition 2.** *Given  $v_n$  an invariant measure for a compatible Markovian chain in the product space  $M^n$  represented as a (row) vector in  $\mathbb{R}^{m^n}$ , then*

$$v_{n-1} = v_n Q_{n-1} \quad (3.3)$$

*is an invariant measure in  $M^{n-1}$  represented as a vector in  $\mathbb{R}^{m^{n-1}}$ .*

*Proof.* Straightforward, since formula (3.3) represents the projection  $(p_1)_*$  in Proposition 1: In fact, each column of  $Q_{n-1}$  acts on a fixed configuration  $(i_2, i_3, \dots, i_n) \in M^{n-1}$ , whose sum with the first parameter  $i_1$  ranging from 1 to  $m$  gives the desired projection.  $\square$

As said before, matrix  $Q_{n-1}$  in formula (3.3) is not unique and a different choice of  $Q_{n-1}$  leads to a different distribution  $v_{n-1}$ . Nevertheless, the choice of  $r = 1$  and  $i_r = 1$  leads to the simplest version given by (3.2).

### 3.2 Back to the main example

We go back to the example of Section 2. We apply Proposition 2 to find precisely at which level  $3 \leq n \leq 6$  the bifurcation occurs given an invariant measure for the  $m$ -point motion.

For sake of notation, we denote by  $v_n^0$  an invariant measure at  $n$ -points for the unperturbed system  $\varphi^0$ , ( $\varepsilon = 0$ ), and by  $v_n^\varepsilon$  an invariant measure of the perturbed system  $\varphi^\varepsilon$ , ( $\varepsilon > 0$ ), respectively. We start with  $n = 6$ . We are going to compare  $v_n^0$  and  $v_n^\varepsilon$ , for different initial ( $n = 6$ ) invariant measures. First we consider the invariant measures of both systems which contain the point 123456. Then, in column representation we have

$$v_6^0 = \begin{pmatrix} 1_{123456} \\ 1_{124365} \\ 1_{213465} \\ 1_{214356} \end{pmatrix}_{6^6 \times 1} \quad \text{and} \quad v_6^\varepsilon = \begin{pmatrix} 1_{123456} \\ 1_{123465} \\ 1_{124356} \\ 1_{124365} \\ 1_{213456} \\ 1_{213465} \\ 1_{214356} \\ 1_{214365} \end{pmatrix}_{6^6 \times 1}$$



where the occurrence of the symbol  $1_{i_1 i_2 i_3 i_4 i_5 i_6}$  in the column vector means that the entry  $(i_1, i_2, i_3, i_4, i_5, i_6)$  is strictly positive while all omitted entries are zero. In addition we always assume that the distributions  $v_6^0$  and  $v_6^\varepsilon$  are uniform (on their respective support).

The projections of the invariant measures can easily be performed along the first coordinate  $(p_1)_*$  as in Proposition 2. It means that, according to formula (3.3), one just has to exclude the first entry of a nonzero entry  $(i_1, \dots, i_r)$ ,  $2 \leq r \leq m$ , in  $v_j^i$ , and rearrange again, if necessary, in such a way that the order in which they appear in the column matrix of the reduced level corresponds to the lexicographic order again.

Hence we generate a sequence of vectors, which represent the invariant distributions,  $v_6^0 \curvearrowright v_5^0 \cdots \curvearrowright v_1^0$  and  $v_6^\varepsilon \curvearrowright v_5^\varepsilon \cdots \curvearrowright v_1^\varepsilon$ , where the  $\curvearrowright$  denotes the application of the procedure of the previous paragraph. This yields

$$\begin{aligned} v_5^0 &= Q_5^T v_6^0 = Q_5^T \begin{pmatrix} 1_{123456} \\ 1_{124365} \\ 1_{213465} \\ 1_{214356} \end{pmatrix}_{6^6 \times 1} = \begin{pmatrix} 1_{13465} \\ 1_{14356} \\ 1_{23456} \\ 1_{24365} \end{pmatrix}_{6^5 \times 1} \curvearrowright v_4^0 = \begin{pmatrix} 1_{3456} \\ 1_{3465} \\ 1_{4356} \\ 1_{4365} \end{pmatrix}_{6^4 \times 1} \\ \curvearrowright v_3^0 &= \begin{pmatrix} 1_{356} \\ 1_{365} \\ 1_{456} \\ 1_{465} \end{pmatrix}_{6^3 \times 1} \curvearrowright v_2^0 = \begin{pmatrix} 1_{56} \\ 1_{65} \end{pmatrix}_{6^2 \times 1} \curvearrowright v_1^0 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1_5 \\ 1_6 \end{pmatrix}. \end{aligned}$$

For the perturbed system, following the same algorithm, we find the invariant measures:

$$\begin{aligned} v_5^\varepsilon &= Q_5^T v_6^\varepsilon = Q_5^T \begin{pmatrix} 1_{123456} \\ 1_{123465} \\ 1_{124356} \\ 1_{124365} \\ 1_{213456} \\ 1_{213465} \\ 1_{214356} \\ 1_{214365} \end{pmatrix}_{6^6 \times 1} = \begin{pmatrix} 1_{13456} \\ 1_{13465} \\ 1_{14356} \\ 1_{14365} \\ 1_{23456} \\ 1_{23465} \\ 1_{24356} \\ 1_{24365} \end{pmatrix}_{6^5 \times 1} \curvearrowright v_4^\varepsilon = v_4^0 \\ \curvearrowright v_j^\varepsilon &= v_j^0 \quad \text{for all} \quad 1 \leq j \leq 3. \end{aligned}$$

This shows that the flows exhibits a 5-point bifurcation. Moreover, we shall compare invariant measures of the 6 point motion of  $\varphi^0$  and  $\varphi^\varepsilon$  in  $M^6$  whose supports contain the point  $(1, 2, 1, 4, 1, 6)$ . Hence,

$$v_6^0 = \begin{pmatrix} 1_{121416} \\ 1_{121315} \\ 1_{212425} \\ 1_{212326} \end{pmatrix}_{6^6 \times 1} \curvearrowright v_5^0 = \begin{pmatrix} 1_{21416} \\ 1_{21315} \\ 1_{12425} \\ 1_{12326} \end{pmatrix}_{6^5 \times 1} \curvearrowright v_4^0 = \begin{pmatrix} 1_{1416} \\ 1_{1315} \\ 1_{2425} \\ 1_{2326} \end{pmatrix}_{6^4 \times 1}$$

$$\curvearrowright v_3^0 = \begin{pmatrix} 1_{416} \\ 1_{315} \\ 1_{425} \\ 1_{326} \end{pmatrix}_{6^3 \times 1} \quad \curvearrowright v_2^0 = \begin{pmatrix} 1_{16} \\ 1_{15} \\ 1_{25} \\ 1_{26} \end{pmatrix}_{6^2 \times 1} \quad \curvearrowright v_1^0 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1_5 \\ 1_6 \end{pmatrix}.$$

And for the perturbed system:

$$v_6^\varepsilon = \begin{pmatrix} 1_{121416} \\ 1_{121415} \\ 1_{121316} \\ 1_{121315} \\ 1_{212426} \\ 1_{212425} \\ 1_{212326} \\ 1_{212325} \end{pmatrix}_{6^6 \times 1} \quad \curvearrowright v_5^\varepsilon = \begin{pmatrix} 1_{12426} \\ 1_{12425} \\ 1_{12326} \\ 1_{12325} \\ 1_{21416} \\ 1_{21415} \\ 1_{21316} \\ 1_{21315} \end{pmatrix}_{6^5 \times 1} \quad \curvearrowright v_4^\varepsilon = \begin{pmatrix} 1_{1416} \\ 1_{1415} \\ 1_{1316} \\ 1_{1315} \\ 1_{2426} \\ 1_{2425} \\ 1_{2326} \\ 1_{2325} \end{pmatrix}_{6^4 \times 1} \quad \curvearrowright v_3^\varepsilon = \begin{pmatrix} 1_{416} \\ 1_{415} \\ 1_{316} \\ 1_{315} \\ 1_{426} \\ 1_{425} \\ 1_{326} \\ 1_{325} \end{pmatrix}_{6^3 \times 1}$$

$$\curvearrowright v_2^\varepsilon = v_2^0 \quad \curvearrowright v_1^\varepsilon = v_1^0.$$

This shows that the flow also exhibits a 3-point bifurcation.

## 4 Degrees of freedom for fixed $k$ -point Markovian characteristics

The purpose of this section is to find formulae for the dimensions of the vector space of distributions of i.i.d. random maps in  $M = \{1, 2, \dots, m\}$  which generate flows in  $M$  with the same prescribed  $k$ -point family of transition probabilities.

Again, as in Section 2, we use the notation  $f_{i_1 \dots i_m}$  for the function

$$(1, 2, 3, 4, 5, 6) \mapsto (i_1, i_2, i_3, i_4, i_5, i_6),$$

with  $i_1, \dots, i_m \in M$ . The stochastic flow of maps  $(\varphi_n)_{n \geq 0}$  in  $M$  is generated by i.i.d. random variables in the space of maps with the following discrete probability distribution

$$\nu = \sum_{i_1, \dots, i_m=1}^m \alpha_{i_1 \dots i_m} \delta_{f_{i_1 \dots i_m}}, \quad (4.1)$$

where  $\delta_{f_{i_1 \dots i_m}}$  is a Dirac measure at the mapping  $f_{i_1 \dots i_m}$ . The non-negative coefficients  $\alpha_{i_1 \dots i_m} \in \mathbf{R}$  are ordered lexicographically by the sub-indices. The first linear restriction on these  $m^m$  coefficients comes from the fact that they determine the distribution of a random variable, hence

$$\sum_{i_1, \dots, i_m=1}^m \alpha_{i_1 \dots i_m} = 1. \quad (4.2)$$

We call this the 0-level restriction for the coefficients. In general, at the  $k$ -level, for a given family of transition probability in  $k$ -point motion  $p_{u_1, \dots, u_k, v_1, \dots, v_k}$ , these characteristics determine linear

restrictions for the coefficients  $\alpha_{i_1, \dots, i_m}$  given by:

$$\sum_{(i_1, \dots, i_{m-k}) \in M^{m-k}} \alpha_{(i_1, \dots, i_{m-k}) \triangleleft \begin{pmatrix} v_1, \dots, v_k \\ u_1, \dots, u_k \end{pmatrix}} = p_{u_1 \dots u_k, v_1 \dots v_k}, \quad (4.3)$$

where the expression  $(i_1, \dots, i_k) \triangleleft \begin{pmatrix} v_1, \dots, v_k \\ u_1, \dots, u_k \end{pmatrix}$  is the shorthand notation for the following vector

$$(i_1, \dots, i_{u_1-1}, v_1, i_{u_1+1}, \dots, i_{u_2-1}, v_2, i_{u_2+1}, \dots, i_{u_k-1}, v_k, i_{u_k+1}, \dots, i_m).$$

Obviously, the degree of freedom (dimension of subspaces which preserve the  $k$ -point characteristics) is given by  $m^m$  minus the number of linearly independent restrictions for the coefficients  $\alpha_{i_1, \dots, i_m}$ . Note that, in particular, the maximal  $m$ -point transition probabilities determines uniquely the coefficients  $\alpha_{i_1, \dots, i_m}$ . In fact, less than that, the information of the transition probabilities of a single element  $(u_1, \dots, u_m)$  with different entries  $u_i$ 's (i.e. it does not belong to any sub-diagonal) is enough to determine all the  $m^m$  coefficients  $\alpha_{i_1, \dots, i_m}$ . In other words: the information of the characteristics at level  $m$  means zero degree of freedom on the choice of the random mappings.

Given the family of transition probability in  $k$ -point motion  $p_{u_1 \dots u_k, v_1 \dots v_k}$ , next result determines the number of linearly independent restrictions for the coefficients  $\alpha_{i_1, \dots, i_m}$ . For a fixed  $m \in \mathbb{N}$ , let  $0 \leq k \leq n \leq m$  and consider the 2-index parameter generated recursively by

$$R_k^n = R_{k-1}^n + \binom{n}{k} (m^k - R_{k-1}^k), \quad (4.4)$$

with  $R_0^n = 1$  for all  $0 \leq n \leq m$ .

**Theorem 2.** *For a finite space  $M = \{1, 2, \dots, m\}$ , given a family of compatible transition probabilities of  $k$ -point motion with  $0 \leq k \leq m$ , the number of linearly independent restrictions for the coefficients  $(\alpha_{i_1 \dots i_m})$  is given by  $R_k^m$ , defined above. In particular,  $R_m^m = m^m$ .*

*Proof.* For any  $0 \leq k \leq n \leq m$ , let  $R_k^n$  denote the number of restrictions (l.i. equations) at  $n$ -point when  $k$ -point characteristics are given. We are going to prove by induction on  $k$  that  $R_k^n$  satisfies the recursive equation (4.4).

Initially, note that when  $k = 0$ , it means that there is no further restrictions other than equation (4.2), then  $R_0^n = 1$  for all  $0 \leq n \leq m$ .

Assume that the formula holds for  $R_{k-1}^n$ , for all  $n \in \{k-1, k, \dots, m\}$ . The restrictions  $R_k^n$  at the  $n$ -level depend also on the characteristics of the  $(k-1)$ , i.e. it is a sum of  $R_{k-1}^n$  plus some new restrictions depending exactly on characteristics at level  $k$ . This justifies the first summand on the right hand side of equation (4.4). We just have to describe this new restrictions depending exactly on characteristics at level  $k$ .

By formula (4.3), considering the projection at level  $k$  from level  $n$  means that there is a subset of positions  $\{\tilde{u}_1, \dots, \tilde{u}_k\} \subseteq \{u_1, \dots, u_n\}$  such that

$$\sum_{(i_1, \dots, i_{m-k}) \in M^{m-k}} \alpha_{(i_1, \dots, i_{m-k}) \triangleleft \begin{pmatrix} v_1, \dots, v_k \\ \tilde{u}_1, \dots, \tilde{u}_k \end{pmatrix}} = p_{\tilde{u}_1 \dots \tilde{u}_k, v_1 \dots v_k}. \quad (4.5)$$

The number of possible subsets with  $k$  elements of  $\{u_1, \dots, u_n\}$  determines the factor  $\binom{n}{k}$  in equation (4.4). Moreover, each parameter  $v_1, \dots, v_k$  has  $m$  different possibilities, hence, expression (4.5) generates a block of  $m^k$  new equations (compare with the detailed example for  $m = 4$  in the Appendix). But many of this equations are linear combination of equations already counted in

$R_{k-1}^n$ , hence this intersection, whose dimension is given by  $R_{k-1}^k$  (by definition) has to be subtracted in each block. This gives the recursive equation (4.4).

In particular, the main result of the theorem is obtained by taking  $n = m$ . Last statement is also trivial by taking  $k = n = m$ .  $\square$

**Example:** One easily verifies that for any  $m \geq 3$  we have that  $m^m - R_2^m > 0$ . It means that in this case one can always find different stochastic flows of mappings which differ on the 3-point motion, but coincide in the levels of 1-point motion and 2-point motion. For  $m = 3$ , for example we have  $R_1^3 = 7$ ,  $R_2^3 = 19$  and  $R_3^3 = 27$ . Hence, there exists an 8-dimensional subspace between 3-point restrictions (of dimension  $R_3^3$ ) and 2-point restrictions (of dimension  $R_2^{3,3}$ ) which is given by distributions in the space of mappings which preserve the characteristics of the 2-point motion.

We present a basis for this 8-dimensional space. For pairwise different  $i, j, k \in M = \{1, 2, 3\}$ , we consider an  $\epsilon$ -perturbation of the distributions in direction of

$$f_{ijk} - f_{jjk} - f_{ikk} - f_{jki} + f_{jji} + f_{jkk} - f_{iji} + f_{iki},$$

Note that the 2-point Markovian dynamics is not affected. Varying  $i, j, k$ , we have that the 6 possible vectors are linearly dependent (since their sum vanishes), but any choice of 5 of these vectors are linearly independent. The remaining three directions of the space of 2-point preserving dynamics can be described by the following vectors: For  $i, j \in M$ ,  $i \neq j$ , consider

$$f_{iii} - f_{iij} + f_{ijj} - f_{iji} + f_{jij} - f_{jii} + f_{jji} - f_{jjj}.$$

$\square$

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## Appendix A: Flows of random permutations of $\{1, \dots, m\}$

In this appendix we gather some properties and give a partial answer for the problem of the number of restrictions on the coefficients for flows of bijections. As before we use the same notation for bijective mappings  $f_{i_1 \dots i_m} : M \rightarrow M$ , but here, we have that  $i_1 \dots i_m$  is a permutation of  $\{1, 2, \dots, m\}$ . The stochastic flow of bijections  $(\varphi_n)_{n \geq 0}$  in  $M$  is generated by i.i.d. random variables in the space of permutations with the following distribution:

$$\nu = \sum_{(i_1, \dots, i_m) \in \text{Per}(\{1, \dots, m\})} \alpha_{i_1 \dots i_m} \delta_{f_{i_1 \dots i_m}} \quad (4.6)$$

The first linear restriction on these  $m!$  coefficients comes from the fact that they determine the distribution of a random variable, hence

$$\sum_{(i_1, \dots, i_m) \in \text{Per}(\{1, \dots, m\})} \alpha_{i_1 \dots i_m} = 1. \quad (4.7)$$

We call this the 0-level restriction for the coefficients. As before, at the  $k$ -level, for a (compatible) family of transition probability in  $k$ -point motion  $p_{u_1 \dots u_k, v_1 \dots v_k}$ , they determine linear restrictions for the coefficients  $\alpha_{i_1, \dots, i_m}$  given by:

$$\sum_{(i_1, \dots, i_{m-k}) \in \text{Per}(\{1, \dots, m\} \setminus \{v_1, \dots, v_k\})} \alpha_{(i_1, \dots, i_{m-k}) \triangleleft (v_1, \dots, v_k)} = p_{u_1 \dots u_k, v_1 \dots v_k} \quad , \quad (4.8)$$

where the sum is taken over  $(m-k)!$  indices. As before, varying  $v_1, \dots, v_k$  in the expression above generates a block of  $m!/(m-k)!$  equations.

In any level  $k$ , the diagonal and its complementary are invariant sets for the dynamics of random permutations. Moreover, for flows of bijections in a finite space, given the sub-maximal  $(m-1)$ -point transition probabilities, they already determine uniquely the maximal  $m$ -point transition probabilities, hence they also determine the  $m!$  coefficients  $\alpha_{i_1 \dots i_m}$ .

Let  $u = (u_1, \dots, u_k)$  and  $v = (v_1, \dots, v_k)$  be elements in  $M^k$ . Since the order of the entries of the elements in  $M^k$  does not matter in a flow, then, if  $\sigma$  is a permutation of  $k$  elements, then the transition probabilities satisfy:

$$p_{u_1 \dots u_k, v_1 \dots v_k} = p_{u_{\sigma(1)} \dots u_{\sigma(k)}, v_{\sigma(1)} \dots v_{\sigma(k)}}.$$

Consider now  $u' = (u'_1, \dots, u'_{(m-k)})$  and  $v' = (v'_1, \dots, v'_{(m-k)})$  elements in  $M^{(m-k)}$  such that, as subsets, they complement  $u$  and  $v$  respectively, i.e.  $\{u\} \cup \{u'\} = \{v\} \cup \{v'\} = M$ . Then

$$\sum_{\sigma \in \Delta} p_{u_1 \dots u_k, v_{\sigma(1)} \dots v_{\sigma(k)}} = \sum_{\xi \in \Delta'} p_{u'_1 \dots u'_k, v'_{\xi(1)} \dots v'_{\xi(k)}}. \quad (4.9)$$

where  $\Delta$  are permutations on  $k$  elements and  $\Delta'$  are permutations in  $(m-k)$  elements. This is obvious from the observation that in a flow of bijections, the whole set  $\{u\}$  is sent to  $\{u'\}$  (independently of the order), if and only if its complementary  $\{v\}$  is sent to  $\{v'\}$ , the complementary of  $\{u'\}$ . For example:

$$p_{1,1} = \sum_{\xi \in \text{Per}(\{2,3,\dots,m\})} p_{2 \dots m, \xi(2)\xi(3)\dots\xi(m)}.$$

The flow of bijections induced in the  $k$ -point level sends each whole fibre (component) into a whole fiber. Again by the Birkhoff-von Neumann theorem the matrix of transition probabilities in  $k$ -point level is again bi-stochastic for all  $1 \leq k \leq m$ . As we have pointed out in the introduction, in our context, when one deals with permutations, it means that one enters in the theory of Birkhoff polytopes, with many open problems. At the moment, we can give only a partial answer for the problem of number of restrictions on the coefficients for a flow of bijections:

**Proposition 3.** *For a finite space  $M = \{1, 2, \dots, m\}$ , given probability transitions of 1-point motion, the number of linearly independent restrictions for the coefficients  $(\alpha_{i_1 \dots i_m})$  is given by*

$$R_1^m = (m-1)^2 + 1.$$

*Proof.* The bi-stochastic  $m \times m$ -matrix of transition probabilities of the 1-point motion has  $2m - 1$  redundancies by definition. These redundancies corresponds to linearly dependent equations of type (4.8) with  $k = 1$ . Hence the restrictions are given by  $[m^2 - (2m - 1)]$  i.e. equations added to the 0-level restriction.  $\square$

The arguments in the proofs of Theorem 2 and of Proposition 3 are not easily extensible to higher levels  $k$  in the case of bijections. This is due to the fact that, in this case, for  $k > 1$  there are further restrictions which involve crossed equations coming from different blocks of equations generated by each fixed  $u_1 u_2 \dots u_k$  in equation (4.8) (in contrast to the previous case of arbitrary self-maps). Moreover, for level  $k \geq m/2$ , new restrictions, coming from equation (4.9) which represents further dependence on lower levels  $(m - k) \leq k$ , arises (again different from the case of self-maps). Therefore, combinatorially, it looks non trivial to control the restrictions coming from different properties with non-empty intersections. As far as our knowledge, in the case of flow of random bijections, the problem of number of restrictions on the coefficients, given the transition probabilities of  $k$ -point motion, for  $k > 1$ , is still open.

## Appendix B: Flow of maps with $m = 4$

In this appendix we illustrate in details the arguments used in the proof of Theorem 2. We consider here flows in  $M = \{1, 2, 3, 4\}$  where there are  $4^4 = 256$  different coefficients  $\alpha_{i_1 i_2 i_3 i_4}$ .

**0-point motion:**

$$\sum_{ijkl} \alpha_{ijkl} = 1$$

Restrictions are given simply by  $R_0^4 = 1$

**1-point motions:** We have  $\binom{4}{1}$  blocks, each block with  $m^1$  new equations:

$$\begin{aligned} \sum_{ijk} \alpha_{uijk} &= p_{1,u}, & u \in M, \\ \sum_{ijk} \alpha_{iujk} &= p_{2,u}, & u \in M, \\ \sum_{ijk} \alpha_{ijuk} &= p_{3,u}, & u \in M, \\ \sum_{ijk} \alpha_{ijk u} &= p_{4,u}, & u \in M. \end{aligned}$$

In each block we have  $R_0^1 = 1$  linearly dependent equations which has to be subtracted from the total number of equations in the block. Hence, linearly independent restrictions are given by:

$$\binom{4}{0} + \binom{4}{1} [4^1 - \binom{1}{0} 4^0] = 1 + 4 * (4 - 1) = 13$$



**2-point motions** We have  $\binom{4}{2}$  blocks, each block with  $m^2$  new equations:

$$\begin{aligned}
\sum_{ij} \alpha_{uvij} &= p_{12,uv}, & u, v \in M, \\
\sum_{ij} \alpha_{iuvj} &= p_{23,uv}, & u, v \in M, \\
\sum_{ij} \alpha_{ijuv} &= p_{34,uv}, & u, v \in M, \\
\sum_{ij} \alpha_{uivj} &= p_{13,uv}, & u, v \in M, \\
\sum_{ij} \alpha_{iujev} &= p_{24,uv}, & u, v \in M, \\
\sum_{ij} \alpha_{uijev} &= p_{14,uv}, & u, v \in M.
\end{aligned}$$

In each block we have  $R_1^2 = 7$  linearly dependent equations (obtained by putting together the reduction from 2-point motion to 1-point and 0-point motion) which has to be subtracted from the total number of equations of the block. Hence, linearly independent restrictions are given by:

$$\begin{aligned}
\binom{4}{0}[4^0] + \binom{4}{1}[4^1 - \binom{1}{0}4^0] + \binom{4}{2}[4^2 - [\binom{2}{0}4^0 + \binom{2}{1}[4^1 - \binom{1}{0}4^0]]] &= 1 + 12 + 6(16 - 1 - 2 * 3) \\
&= 13 + 6 * 9 \\
&= 67.
\end{aligned}$$

Remaining degrees of freedom  $4^4 - 67 = 189$ .

**3-point motions** We have  $\binom{4}{3}$  blocks, each block with  $m^3$  new equations:

$$\begin{aligned}
\sum_i \alpha_{uvwi} &= p_{123,uvw}, & u, v, w \in M, \\
\sum_i \alpha_{uviw} &= p_{124,uvw}, & u, v, w \in M, \\
\sum_j \alpha_{uivw} &= p_{134,uvw}, & u, v, w \in M, \\
\sum_i \alpha_{iuvw} &= p_{234,uvw}, & u, v, w \in M.
\end{aligned}$$

In each block we have  $R_2^3 = 37$  linearly dependent equations (obtained by putting together the reduction from 3-point motion to 2-point, 1-point and 0-point motion) which has to be subtracted from the total number of equations of the block. Hence, linearly independent restrictions are given by:

$$\begin{aligned}
& \binom{4}{0}[4^0] + \\
& \binom{4}{1}[4^1 - \binom{1}{0}[4^0]] + \\
& \binom{4}{2}[4^2 - (\binom{2}{0}4^0 + \binom{2}{1}[4^1 - \binom{1}{0}[4^0]])] + \\
& \binom{4}{1}[4^3 - (\binom{3}{0}4^0 + \binom{3}{2}[4^1 - \binom{1}{0}[4^0]] + \binom{3}{1}[4^2 - (\binom{2}{0}4^0 + \binom{2}{1}[4^1 - \binom{1}{0}[4^0]])])] \\
& = 67 + 4[64 - [10 + 3 * 9]] = 175.
\end{aligned}$$

**4-point motions** We have a single  $\binom{4}{4} = 1$  block, with  $m^4$  new equations:

$$\alpha_{uvwx} = p_{1234,uvwx},, \quad u, v, w, x \in M,$$

In this single block, we have  $R_3^4 = 175$  linearly dependent equations (obtained by putting together the reduction from 4-point motion to 3-point, 2-point, 1-point and 0-point motion) which has to be subtracted from the total number of equations of the block. Hence, linearly independent restrictions are given by the following equation, which one easily sees that has a telescopic cancellation:

$$\begin{aligned}
& \binom{4}{0}[4^0] + \\
& \binom{4}{1}[4^1 - \binom{1}{0}[4^0]] + \\
& \binom{4}{2}[4^2 - (\binom{2}{0}4^0 + \binom{2}{1}[4^1 - \binom{1}{0}[4^0]])] + \\
& \binom{4}{3}[4^3 - (\binom{3}{0}4^0 + \binom{3}{1}[4^1 - \binom{1}{0}[4^0]] + \binom{3}{2}[4^2 - (\binom{2}{0}4^0 + \binom{2}{1}[4^1 - \binom{1}{0}[4^0]])])] + \\
& \binom{4}{4}[4^4 - (\binom{4}{0}[4^0] + \binom{4}{1}[4^1 - \binom{1}{0}[4^0]] + \binom{4}{2}[4^2 - (\binom{2}{0}4^0 + \binom{2}{1}[4^1 - \binom{1}{0}[4^0]])] \\
& + \binom{4}{3}[4^3 - (\binom{3}{0}4^0 + \binom{3}{1}[4^1 - \binom{1}{0}[4^0]] + \binom{3}{2}[4^2 - (\binom{2}{0}4^0 + \binom{2}{1}[4^1 - \binom{1}{0}[4^0]])])] \\
& = 4^4.
\end{aligned}$$

We finish this appendix with the numbers for  $m = 5$ , just for comparison, without details. We have  $R_1^5 = 21$ ,  $R_2^5 = 181$ ,  $R_3^5 = 821$ ,  $R_4^5 = 2101$  and  $R_5^5 = 5^5 = 3125$ .